Estimation of the Volterra Kernels of a Nonlinear System Using Impulse Response Data

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Abstract

A method of identification of nonlinear dynamic systems, based on the models constructed as Volterra series with the use of pulse test signals, is proposed. To separate the response into partial components, the method based on differentiation of the target output signal with respect to the parameter – amplitude is applied. The errors of the identification method are analyzed. By the example of modeling a system with nonlinear feedback, the dependences of identification errors on the test signal amplitude are obtained. To increase the noise stability of the identification method, the smoothing of estimations of the Volterra kernels with the help of the wavelet transforms is used.

1. Introduction

In describing nonlinear dynamic systems and circuits the mathematical models in the form of Volterra integro-power series [1] find their increasing application. However, current applied algorithms of identification of nonlinear systems based on the Volterra series [2] do not yet allow using this mathematical tool to the full. It is caused by a series of reasons, most important of which is the essential influence of errors of measurements on the result of identification [3]. Another problem is that the identification of nonlinear systems as Volterra series includes the separation of the response of the system under study into partial components corresponding to separate terms of the Volterra series with subsequent determination of the multidimensional weight functions (Volterra kernels).

One of the ways of the separation consists in composing linear combinations of the responses to the test signals of various amplitudes [4]. In paper [5] the values of the amplitudes of the test influences and corresponding coefficients were obtained, allowing to minimize the methodical error of the model splitting into partial components, caused by the influence of the Volterra series terms which order is higher than the determined term. In paper [3] the ways of identification based on the methods of regularization of ill-posed problems have been proposed.

In the present work we propose a method of identification of nonlinear systems in the form of the Volterra series, based on the extraction of partial components of the system response with the help of differentiation of the target output signal with respect to the parameter – amplitude. The errors of the identification method are analyzed. By the example of modeling a system with nonlinear feedback, we obtain the dependences of the identification errors on the test signal amplitude. To increase the noise stability of the identification method we apply the smoothing of estimations of the Volterra kernels using of the wavelet transforms [6, 7].

2. On the Use of Volterra Series for Identification of Nonlinear Systems

In the general case the "input-output" relationship for a nonlinear dynamic object can be represented in terms of the Volterra series as

$$y[x(t)] = \sum_{n=1}^{\infty} y_n(t) = \sum_{n=1}^{\infty} \int_0^{\infty} \ldots \int w_n(\tau_1, \tau_2, \ldots, \tau_n) \prod_{i=1}^{n} x(t-\tau_i) d\tau_i, \quad (1)$$

where $x(t)$ and $y[x(t)]$ are the input and output signals, respectively, $w_n(\tau_1, \tau_2, \ldots, \tau_n)$ is the Volterra kernel of the n-th order and $y_n(t)$ stands for the n-th partial component of the object response.

Commonly, the Volterra series are replaced by a polynomial, with only taking several first terms of series (1) into consideration. Then the identification procedure consists in extracting the partial components with subsequent determination of Volterra kernels $w_n(\tau_1, \tau_2, \ldots, \tau_n)$.

Expanding function $f_k(x_1, x_2, \ldots, x_k)$ into the Taylor series in the vicinity of point $x^0 = (x_1^0, x_2^0, \ldots, x_k^0)$, one has:

$$f_k(x_1, x_2, \ldots, x_k) = f_k(x_1^0, x_2^0, \ldots, x_k^0) + \sum_{l=1}^{n} \left.\frac{\partial f_k}{\partial x_l}\right|_{x^0} \Delta x_l + \frac{1}{2!} \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} \left.\frac{\partial^2 f_k}{\partial x_{l_1} \partial x_{l_2}}\right|_{x^0} \Delta x_{l_1} \Delta x_{l_2} + \ldots, \quad (2)$$

where $\Delta x_l = x_l - x^0$.

We henceforth consider the case $x^0 = 0$ and $f_k(x_1, x_2, \ldots, x_k) = 0$, i.e. it is supposed that prior to signal injection the object is at rest (zero initial conditions). Function f_k depends also on parameter t, i.e. $f_k(t, x_1, x_2, \ldots, x_k)$, so that expression (2) can be written as

$$F[x_k(\tau), 0 \le \tau \le t] = \sum_{l=1}^{k} w_1(t, \tau_l) \Delta x_l \Delta \tau_l + \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} w_2(t, \tau_{l_1}, \tau_{l_2}) \Delta x_{l_1} \Delta x_{l_2} \Delta \tau_{l_1} \Delta \tau_{l_2} + \ldots, \quad (3)$$

where $\Delta\tau_l = \tau_l - \tau_{l-1} = \Delta\tau, \quad \Delta\tau = \frac{t}{k}$;

$$w_n(t, \tau_{l_1}, \ldots, \tau_{l_n}) = \frac{1}{n!(\Delta\tau)^n} \left.\frac{\partial^n f_k}{\partial x_{l_1} \ldots \partial x_{l_n}}\right|_{x=0}. \quad (4)$$

In the limit $\Delta\tau \to 0 (k \to \infty)$ Eq.(3) turns into series (1).

2. Estimation of Volterra kernels

We use the method of extracting the partial components with the help of n-fold differentiation of the response $y(a, t)$ with respect to parameter - amplitude a and the use of the derivative value at a=0 [5].

Injecting an input signal $ax(t)$ where a is the scaling factor (signal amplitude), one has the following response of the nonlinear system:

$$y[a \cdot x(t)] = a\int_0^t w(\tau) \cdot x(t-\tau)d\tau +$$
$$+a^2 \int_0^t \int_0^t w_2(\tau_1, \tau_2) \cdot x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 + \quad (5)$$
$$+a^n \underbrace{\int_0^t \dots \int_0^t}_{n\ time} w_n(\tau_1, ..., \tau_n) \prod_{r=1}^{n} x(t-\tau_r)d\tau_r + ...$$

To distinguish the partial component of the n-th order, differentiate the system response n times with respect to the amplitude:

$$\frac{\partial^n y[a \cdot x(t)]}{\partial a^k} = n! \underbrace{\int_0^t \dots \int_0^t}_{n\ time} w_n(\tau_1, ..., \tau_n) \prod_{r=1}^{n} x(t-\tau_r)d\tau_r +$$
$$+(n+1)! \cdot a \underbrace{\int_0^t \dots \int_0^t}_{n+1\ time} w_{n+1}(\tau_1, ..., \tau_{n+1}) \prod_{r=1}^{n+1} x(t-\tau_r)d\tau_r + ... \quad (6)$$

Taking the value of the derivative at a=0, we finally obtain the expression for the partial component:

$$y_n(t) = \underbrace{\int_0^t \dots \int_0^t}_{n\ time} w_n(\tau_1, ..., \tau_n) \prod_{r=1}^{n} x(t-\tau_r)d\tau_r = \quad (7)$$
$$= \frac{1}{n!} \frac{\partial^n y[a \cdot x(t)]}{\partial a^n}\bigg|_{a=0}$$

Given the function in the discrete form, the differentiation is performed numerically. The corresponding formulae for the first and second derivatives in finite (equidistant) differences read [6]:

$$y'_0 = \frac{1}{2a}(-y_{-1} + y_1),$$
$$y'_0 = \frac{1}{12a}(y_{-2} - 8y_{-1} + 8y_1 + y_2),$$
$$y''_0 = \frac{1}{a^2}(y_{-1} - 2y_0 + y_1),$$
$$y''_0 = \frac{1}{12a^2}(-y_{-2} + 16y_{-1} - 30y_0 + 16y_1 - y_2),$$
$$y'''_0 = \frac{1}{2a^3}(-y_{-2} + 2y_{-1} - 2y_1 + y_2). \quad (8)$$

where

$$y'_0 = y'(0),\ y''_0 = y''(0),\ y_i = y(ia),\ i = -2, -1, 0, 1, 2$$

To find the Volterra kernel of the 2[nd] order we use the third and fourth formulae in (8). They allow us to find the values of the second derivative in node y_0 using the values of the function in nodes y_{-2}, y_{-1}, y_0, y_1, y_2. Actually, since the zeroth node value (at a=0) of the function is equal to zero, for extracting the 2[nd] order partial component it is necessary to conduct two experiments minimum, injecting signals of the same form and amplitude but of different polarity.

After extracting partial component $y_n(t)$ and applying additional processing one can determine the section of the Volterra kernel of the n-th order.For the diagonal section we have [7]:

$$\hat{w}_n(t, t, ..., t) = \frac{y_n(t)}{(\Delta\tau)^n}, \quad n = 1, 2, ..., \quad (9)$$

where $\hat{w}_n(t, t, ..., t)$ is the estimation of the the diagonal section of the Volterra kernel of the n order and $\Delta\tau$ stands for the test pulse duration.

For the lateral sections of multidimensional Volterra kernels of a nonlinear object we have the following approximate expression:

$$\hat{w}_n(t-t_1, ..., t-t_n) = \frac{(-1)^n}{n!(\Delta\tau)^n} \sum_{\delta_{t_1}, ..., \delta_{t_n}=0}^{1} (-1)^{\sum_{i=1}^{n}\delta_{t_i}} y(t, \delta_{t_1}, ..., \delta_{t_n}). \quad (10)$$

where $\hat{w}_n(t-t_1, ..., t-t_n)$ is the estimation of the lateral section of the Volterra kernel of the n-th order, obtained as a result of processing experimental data, $y\ (t, \delta_{t_1}, ..., \delta_{t_n})$ is the system reaction measured at instant t provided that the injected delta-like pulses are of amplitude a and duration $\Delta\tau$ and applied at instants $t_1,..,t_n$, respectively ($\delta_t = 1$ corresponds to an injected pulse at instant t while $\delta_t = 0$ means no injected pulse).

3. Simulations

To analyze the identification method, its accuracy and noise stability by computer modeling within the package of applied programs MATLAB we choose an object described by the following nonlinear differential equation:

$$\frac{dy(t)}{dt} + \alpha \cdot y(t) + \beta \cdot y^2(t) = x(t) \quad (11)$$

where α and β are constant coefficients (α=2.64 and β=1.45). For such an object the model in the form of three terms of the Volterra kernel under zero initial conditions reads:

$$y(t) = \int_0^t w_1(\tau_1)x(t-\tau_1)d\tau_1 + \int_0^t\int_0^t w_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 +$$
$$+\int_0^t\int_0^t\int_0^t w_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 \tau_3. \quad (12)$$

The three first weight functions $w_n(\tau_1, \tau_2, ..., \tau_n)$ for the given object are:

$$w_1(\tau_1) = e^{-\alpha\tau_1},\quad w_2(\tau_1,\tau_2) = \frac{\beta}{\alpha}(e^{-\alpha(\tau_1+\tau_2)} - e^{-\alpha\tau_2}),\quad \tau_1 \le \tau_2,$$
$$w_3(\tau_1,\tau_2,\tau_3) = \frac{1}{3}\left(\frac{\beta}{\alpha}\right)^2 \cdot (e^{a(\tau_1-\tau_2-\tau_3)} + 3e^{-a(\tau_1+\tau_2+\tau_3)} -$$
$$-4e^{-\alpha(\tau_2+\tau_3)} - 2e^{-\alpha(\tau_1+\tau_3)} + 2e^{-\alpha\tau_3}),\quad \tau_1 \le \tau_2 \le \tau_3. \quad (13)$$

Setting $\tau_1=\tau_2=\tau_3=t$, we obtain the diagonal sections of the Volterra kernels of the 2nd and 3rd orders:

$$w_2(t,t)=\frac{\beta}{\alpha}(e^{-2\alpha t}-e^{-\alpha t}),$$

$$w_3(t,t,t)=\left(\frac{\beta}{\alpha}\right)^2\cdot(e^{-3at}-2e^{-2\alpha t}+e^{-\alpha t}). \tag{14}$$

In estimating the errors of modeling of the diagonal sections of the Volterra kernels we used the root-mean-square error criterion:

$$\varepsilon=\sqrt{\frac{1}{p}\sum_{t=1}^{p}(w_t-\hat{w}_t)^2} \tag{15}$$

where p is the number of counts in the observation time interval, w_t is the exact value of the Volterra kernel and $\hat{w}_t$ is the value of the Volterra kernel estimation obtained by processing experimental data (system responses) at discrete instants t's.

In reality the target output signal is measured with some error, and one can consider it as a superposition of the target signal itself and white noise (errors of measurements, scaled as 1, 3, and 5% of the maximal value of the response).

Table 1 contains the optimal areas (i.e. minimizing the identification error ε) of signals under identification of the Volterra kernels of the second and third order.

Table 1: Optimal areas of input pulse actions (*S*)

n	Optimal areas of input pulse actions (*S*) at various response measurement errors (*eps*):		
	1 %	3 %	5 %
2	0.70	0.9	1.1
3	1.02	1.30	1.38

Figs. 1,2 show the dependences of the identification error (ε) on the area of the input pulse actions (*S*) in determination of the diagonal sections of the Volterra kernels of the second and third orders, respectively, at different errors of measurements of the response (*eps*).

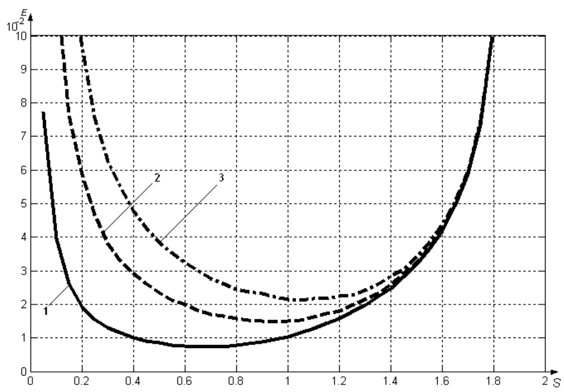

Figure 1: The dependence of the identification error (ε) on the area of the input pulse actions (*S*) in determination of the diagonal sections of the Volterra kernels of the second order. *1,2,3* – 1%, 3%, 5% measurement errors, respectively.

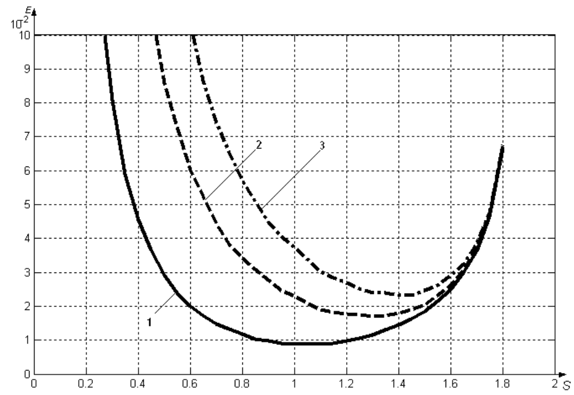

Figure 2: The dependence of the identification error (ε) on the area of the input pulse actions *(S)* in determination of the diagonal sections of the Volterra kernels of the third order. *1,2,3* – 1%, 3%, 5% measurement errors, respectively.

Figs. 3, 4, 5 represent the results of identification of the diagonal section of the Volterra kernels of the second order (*n=2*) at response measurement errors (*eps*) of 1, 3, 5%, respectively, with the areas of the input pulses taken from Table 1. Fig. 6 shows the same for *n=3* and *eps*=1%.

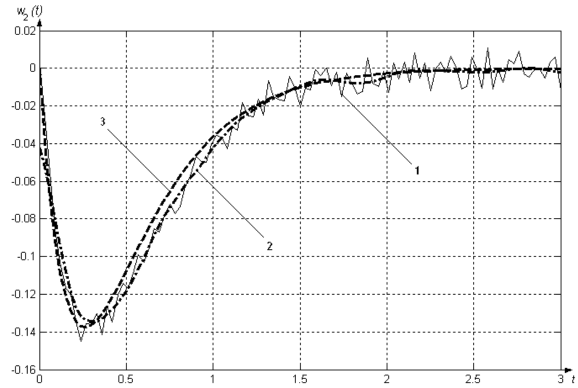

Figure 3: The result of identification of the diagonal sections of the Volterra kernels of the second order. *1* – results of identification, *2* – application of wavelet transforms, *3* – exact value of the Volterra kernel; *eps*=1%.

In Table 2 the identification errors for *n*=2, 3 are given.

Table 2: Identification errors (ε) for *n*=2, 3

n	Identification errors (ε) at various measurement errors					
	1 %	3 %	5 %	1 %	3 %	5 %
	without application of wavelet transform			with application of wavelet transform		
2	0.0085	0.0140	0.0219	0.0055	0.0077	0.0094
3	0.0096	0.0174	0.0235	0.0061	0.0096	0.0112

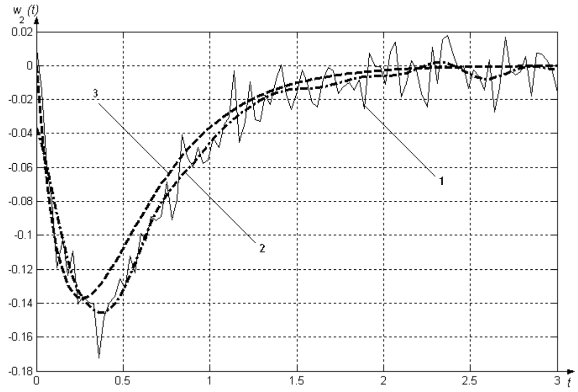

Figure 4: The result of identification of the diagonal sections of the Volterra kernels of the second order. *1* – results of identification, *2* – application of wavelet transforms, *3* – exact value of the Volterra kernel; *eps*=3%.

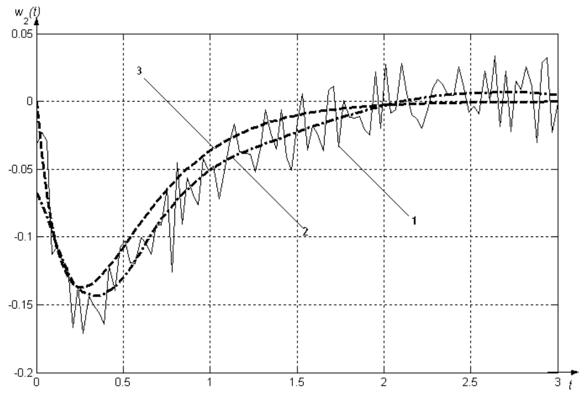

Figure 5: The result of identification of the diagonal sections of the Volterra kernels of the second order. *1* – results of identification, *2* – application of wavelet transforms, *3* – exact value of the Volterra kernel; *eps*=5%.

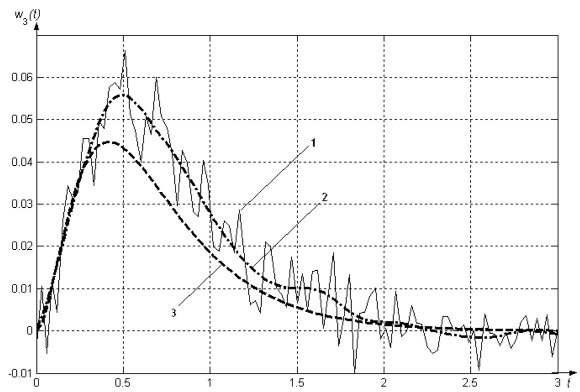

Figure 6: The result of identification of the diagonal sections of the Volterra kernels of the third order. *1* –results of identification, *2* – application of wavelet transforms, *3* – exact value of the Volterra kernel; *eps*=1%.

4. Conclusion

The use of the mathematical models based on the Volterra integro-power series for identification of nonlinear dynamic systems is one of the long-standing problems of the control theory. When determining the multidimensional weight functions (Volterra kernels), however, the problems arise in separating the n-th order partial components from the measured system response to a given perturbation and then in determining the n-dimensional Volterra kernel. Solving these problems is computationally unstable, and this leads to significant identification errors even at small deviations (measurement noises) of initial input data.

We have investigated the errors of identification of a nonlinear system in the form of the Volterra series with the use of testing pulse signals, basing on the separation of the partial components by differentiation of the system response with respect to the parameter-amplitude. Computer experiments (within MATLAB) on the choice of the test signal amplitude are performed and the results of identification of the Volterra kernels of the 1st, 2nd and 3rd orders are presented.

Application of the noise suppression procedure based on the wavelet transformation to estimations of the Volterra kernels allows us to obtain smooth solutions and to lower the identification error 1.5÷3 times. The presented dependences of identification errors on the area of testing actions in determining the diagonal sections of the Volterra kernels allow us to specify the range of optimal amplitudes of pulse actions for different levels of response measurement errors which correspond to the minimal errors of identification of the Volterra kernels.

5. References

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