Statistical characteristics of nonlinear filter outputs for Poisson distributed processes

Pavel Ye. Yeltsov[a], *Alexander N. Besedin*[a], *Sari Peltonen*[b], *Vladimir V. Lukin*[a]

[a] Department of Transmitters, Receivers and Signal Processing, National Aerospace University, Chkalova Str. 17, 61070, Kharkov, Ukraine, Telephone: +38 (0572) 7074841, E-mail: lukin@xai.kharkov.ua
[b] Tampere University of Technology, Institute of Signal Processing, P. O. Box 553, FIN-33101, Tampere, Finland, Telephone: +358 3 3115 3859, Fax: +358 3 3115 3857, E-mail: sari.peltonen@tut.fi

Abstract

The reasons for applying nonlinear filters to Poisson distributed processes are discussed. The statistical characteristics of several scanning window order statistic filter outputs are derived and analyzed for different parameters of input processes and filters. Basic dependencies are obtained analytically and by numerical simulations and then studied. In particular, it is demonstrated that nonlinear filters produce biased outputs.

1. Introduction

The Poisson distribution of noise or parameter estimates arises in quite many situations where the principle of sensor operation is such that it counts a number of occurrences of event within a specified period of time [1]. Counting of calls via switchboard or defects, evaluation of frequency of phase modulated signal parameters, and quantum noise in X-ray images are a few traditional examples [1,2].

Because of the aforementioned principle of sensor operation, the observed 1-D or 2-D processes appear to be distorted by Poisson noise that possesses some specific peculiarities including its non-Gaussianity, asymmetric probability density function (pdf), etc. Since Poisson distributed noise is rather intensive, the corresponding processes require filtering in order to reduce degrading influence of the noise. On the other hand, many of commonly used filters are not well suited for the considered application [2,3]. The majority of signal/image processing methods are designed under assumption of additive and, commonly, Gaussian noise presence. Less number of techniques is intended for processing data corrupted by multiplicative noise. But Poisson distributed noise is signal-dependent and it is neither purely multiplicative nor additive.

The choice of a proper filter becomes especially crucial if the data, in addition to Poisson noise, can also be corrupted by impulsive noise. Presence of impulsive noise forces the use of nonlinear filters that possess robust properties. However, their application and properties for such situations are not intensively studied. For design of effective signal/image processing methods and systems, it is necessary to know the statistical characteristics of nonlinear filter outputs which depend on Poisson distribution and filters parameters. Thus, below we present the results of statistical characteristic analysis for outputs of several typical nonlinear filters such as standard median and α-trimmed ones and consider the basic dependencies for them.

2. Poisson distributed processes

For simplicity, consider 1-D case. Suppose one deals with the sampled information process $S(t_i) = S(i),\ i = 1,...,I,$ (where I is the number of samples) for which each measurement (data sample) obeys Poisson distribution defined by the pdf

$$f(k) = \begin{cases} \dfrac{e^{-\lambda}\lambda^k}{k!} & \text{for integer value } k \ge 0 \\ 0 & \text{otherwise} \end{cases}, \qquad (1)$$

where λ is a real value satisfying the condition $\lambda > 0$. In other words, the true values of the information component of the process are equal to $\lambda(t_i),\ i = 1,...,I$.

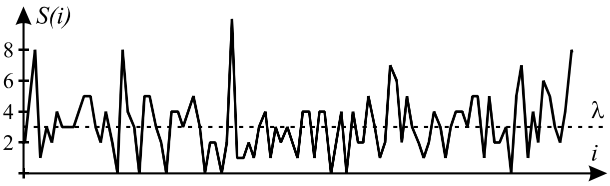

Figure 1.

An example of such process behavior for constant $\lambda = 3$ is shown in Fig. 1. Obviously, quite large variations from the true value are observed. This is explained by the fact that for Poisson pdf for the mean m equal to λ the standard deviation σ is equal to $\sqrt{\lambda}$. If the accuracy of initial measurements is characterized by the ratio $\sigma / m = 1/\sqrt{\lambda}$, then for rather small λ this ratio is quite large and, thus, the accuracy of measurements occurs to be poor. Therefore, it is worth improving and this can be done by means of process filtering.

Other peculiarities of Poisson pdf are the following. First, according to the nature of measurements the process values can be only integer ones. Second, for rather small λ the pdf is considerably asymmetrical with respect to λ (an example is demonstrated in Fig. 2 for λ=3). When λ is large (tens and hundreds) it becomes more symmetrical and approaches sampled (integer valued) Gaussian distribution.

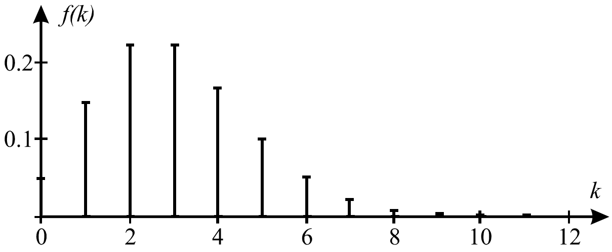

Figure 2.

It can be expected that these properties can lead to specific features of the processed data if nonlinear (order statistic based) filters are applied. In particular, the median filter application produces output bias when applied to processing data with non-Gaussian pdf. An example in favor of this is the filtering of images corrupted by speckle with Rayleigh or one-side exponential pdf [4].

Besides, it is desirable to evaluate noise suppression efficiency of nonlinear filters in order to predict the improvement of parameter estimation accuracy due to processing of $\{S(t_i)\}$ with aforementioned statistical characteristics.

3. Order Statistic Probabilities

According to [5], the probability density function of the *j*-th order statistic for the sample of size *N* of Poisson distributed data is

$$f_{j,N}(x) = \sum_{r=j}^{N} \binom{N}{r} \Big[\{F(x)\}^r \{1 - F(x)\}^{N-r} - \{F(x-1)\}^r \{1 - F(x-1)\}^{N-r} \Big], \quad (2)$$

where $x = 0,1,2,\dots,$ $\binom{N}{r}$ are the binomial coefficients and *F(x)* is the cumulative probability distribution function of the input data. Similarly, the distribution function of the *j*-th order statistic is [5,6]

$$F_{j,N}(x) = \sum_{r=j}^{N} \binom{N}{r} \{F(x)\}^r \{1 - F(x)\}^{N-r} . \quad (3)$$

For the Poisson distribution with λ=3 and *N*=5 the obtained plots of probability densities (2) for all 5 order statistics are given in Fig. 3 and the plots of $F_{j,N}$ are presented in Fig. 4. As can be seen, in opposite to Gaussian distribution [7], the function $f_{j,N}$ for *j*=3 (i.e., for the sample median) is not symmetrical with respect to the distribution mean.

The analytical expressions (2) and (3) permit to easily calculate the mean and variance for all order statistics for given *N*. If *N* is the size of the nonlinear filter scanning window, then one can obtain the statistical characteristics of the corresponding filter output for the signal fragments where $\lambda(t_i) = Const$.

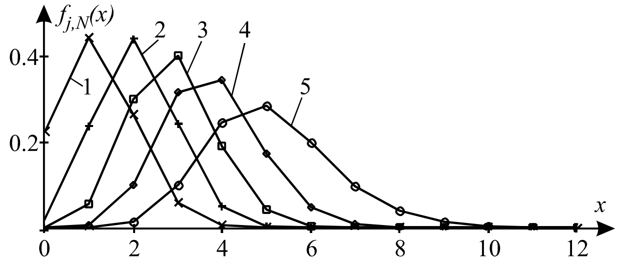

Figure 3.

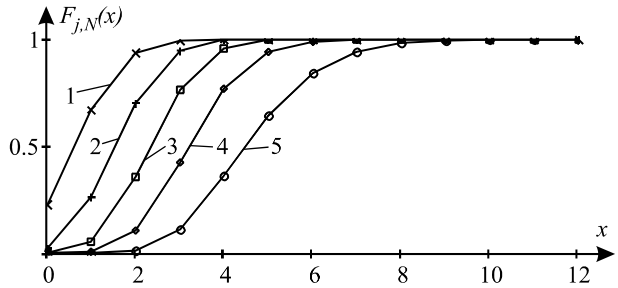

Figure 4

4. Median Filter Output Characteristics

First of all, consider the median filter output. Fig. 5 presents the data obtained analytically. The output bias for median (the order statistic with $j_{med} = (N+1)/2$) was derived as

$$\Delta = \lambda - \int_0^\infty x f_{j_{med},N}(x)dx = \lambda - \sum_{k=0}^{K\to\infty} k\, f_{j_{med},N}(k) . \quad (4)$$

This expression takes into account that the median filter output values coincide with one value of the input sample and, thus, they can only be integer. The parameter *K* in derivations should be finite and large enough. For small $\lambda(t_i)$ (up to 10) the required accuracy is provided if *K*≥50, for larger $\lambda(t_i)$ our recommendation is to set *K*≥3 $\lambda(t_i)$.

As can be seen, the median filter output is biased. The bias depends upon the filter scanning window size *N* and commonly is larger for larger *N*. At the same time, for $\lambda(t_i) \geq 5$ the bias Δ is practically independent of λ. However, the ratio Δ/λ increases when λ decreases. This means that relative systematic error induced by median filter is the most considerable for the signal fragments with rather small $\lambda(t_i)$.

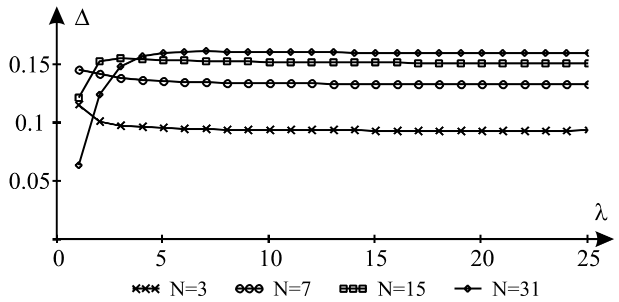

Figure 5.

Let us now consider noise suppressing properties of the standard median filter. They can be characterized by parameter $\chi = \lambda/\sigma_{res}^2$ where σ_{res}^2 is the variance of residual fluctuations (noise) at the filter output. For the standard median filter σ_{res}^2 can be evaluated analytically for given N and λ as

$$\sigma_{res}^2 = \sum_{k=0}^{K\to\infty} k^2 f_{j_{med},N}(k) - \left(\sum_{k=0}^{K\to\infty} k f_{j_{med},N}(k)\right)^2 \qquad (5)$$

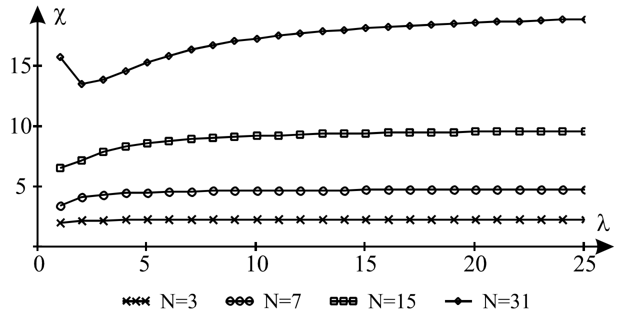

Figure 6.

In Fig. 6 χ is plotted with respect to λ. As expected, χ increases when N becomes larger. It also becomes slightly larger with larger λ. The analysis of the plots in Fig. 6 shows that, in general, for "predictive" or approximate calculation of χ for the standard median filter, in case of processing Poisson distributed processes for large λ, one can use the same expression [3] that is commonly used for Gaussian noise, i.e. $\chi = 2(N + \pi/2 - 1)/\pi$.

The median filter output for the signal presented in Fig. 1 is shown in Fig. 7 for N=15. All the aforementioned peculiarities are observed.

It is also worth noting that the plots very close to those ones represented in Figures 5 and 6 were obtained by numerical simulations performed by averaging the median filter outputs for 10000 samples. The software was designed using MathCAD.

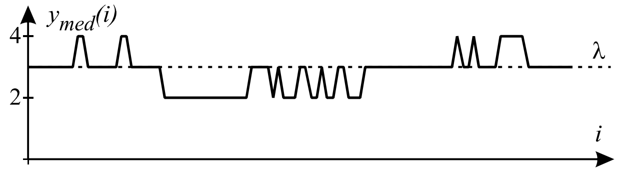

Figure 7.

5. Statistical characteristics of α-trimmed filter output

Recall that the α-trimmed filter output is defined as

$$y_\alpha(i) = (1/(N - N_{\alpha1} - N_{\alpha2})) \sum_{j=N_{\alpha1}+1}^{N-N_{\alpha2}} S_i^{(j)} , \qquad (6)$$

where $N_{\alpha1}, N_{\alpha2}$ are the numbers of minimal and maximal value samples (order statistics) that are rejected from the ordered sample, $S_i^{(j)}$ defines the j-th order statistic for the data taken from the scanning window centered on the i-th sample of the filtered process (sequence).

Commonly $N_{\alpha1} = N_{\alpha2}$, although the use of non-equal trimming parameters can produce some benefits [8] in cases of non-symmetrical pdfs of noise. Since the number of possible combinations of $N_{\alpha1}, N_{\alpha2}$ for the case $N_{\alpha1} \neq N_{\alpha2}$ is too large, especially for large N, let us consider below only particular case $N_{\alpha1} = N_{\alpha2}$ and N=15. Since the statistical characteristics of the α-trimmed filter output also depend upon trimming parameters [3,8], four values $N_{\alpha1} = N_{\alpha2}$ have been tested, namely 2,3,5, and 7 (the latter case is the standard median filter).

For the α-trimmed filter it is possible to evaluate the output mean (if $\lambda(t_i) = Const$) in the following way [3]:

$$m_\alpha = (1/(N - N_{\alpha1} - N_{\alpha2})) \sum_{j=N_{\alpha1}+1}^{N-N_{\alpha2}} m_j , \qquad (7)$$

where $m_j = \int_0^\infty x f_{j,N}(x)dx = \sum_{k=0}^{K\to\infty} k f_{j,N}(k)$.

Then one can also calculate the output bias $\Delta_\alpha = \lambda - m_\alpha$. The obtained dependencies are presented in Fig. 8. As can be seen, these dependencies are very similar to those ones presented in Fig. 5. The α-trimmed filter output is biased and Δ_α depends upon $N_{\alpha1} = N_{\alpha2}$. If $N_{\alpha1} = N_{\alpha2}$ are selected larger, the bias decreases and is smaller than for the standard median filter.

Similar dependencies have also been obtained for other N and $N_{\alpha 1} = N_{\alpha 2}$. For given (fixed) ratios $N_{\alpha 1} / N = N_{\alpha 2} / N$ they practically coincide for different N.

For $\lambda(t_i) \geq 5$ the bias Δ_α is practically independent of λ for given trimming parameters. Again, the ratio Δ_α / λ increases when λ reduces. Thus, one can expect the largest relative systematic error of α-trimmed filter output for signal fragments with rather small $\lambda(t_i)$.

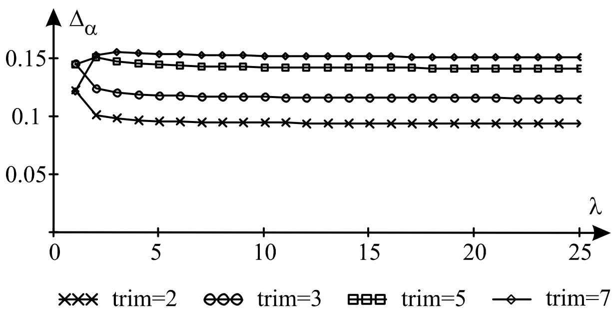

Figure 8.

It is quite difficult to evaluate the noise suppressing properties of α-trimmed filter output analytically since such derivations require numerical integration for estimation of order statistic correlation matrix [3]. Because of this and taking into account the appropriate accuracy of the results obtained by simulations, the parameter χ was estimated numerically. The obtained dependencies are represented in Fig. 9. As can be expected, with reducing $N_{\alpha 1} = N_{\alpha 2}$ the efficiency χ increases and it is better in comparison to the standard median filter with the same N.

Finally, Fig. 10 presents the α-trimmed filter output for the input signal in Fig. 1. Although the α-trimmed filter output also falls into discrete values, they are not only integer like in standard median filter case (compare the plots in Figures 10 and 7).

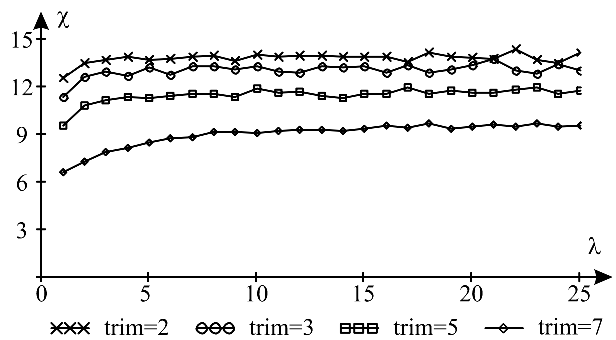

Figure 9.

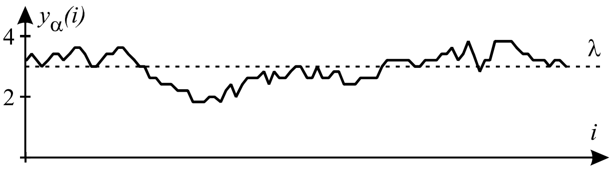

Figure 10.

6. Discussion and conclusions

It is shown that for Poisson distributed processes the nonlinear filter outputs can be biased and this obstacle should be taken into account while designing or selecting the method for such data processing. Analytical expressions and basic dependencies for obtaining and predicting the statistical characteristics of the standard median and α-trimmed filter outputs are given. They can be useful in design of more complex algorithms for filtering of Poisson distributed processes.

References

[1] The Communication Handbook, Second Edition, Edited by J.D. Gibson, CRC Press, Boca Raton, 2002, 896 p.

[2] T. Aach, D. Kunz, Multiscale Linear/Median Hybrid Filters for Noise Reduction in Low Doze X-Ray Images, Proceedings of ICIP, 1997, Vol. 2, pp. 358-361.

[3] Astola J., Kuosmanen P. Fundamentals of nonlinear digital filtering. Boca Raton (USA): CRC Press LLC, 1997, 276 p.

[4] Lukin V.V., Melnik V.P., Pogrebniak A.B., Zelensky A.A., Techniques and Algorithms of Speckle Noise Reduction for One-Look SAR Images, Proceedings of EUSAR'96, Konigswinter (Germany), 1996, pp. 167-170.

[5] Arnold B.C., Balakrishnan N., Nagaraja H.N., A First Course in Order Statistics, A Wiley-Interscience Publication, NY (USA), 1992, 279 p.

[6] Hampel F.R., Ronchetti E.M., Rousseeuw P.J., Stahel W.A., Robust Statistics. The Approach Based on Influence Functions. John Wiley and Sons, NY (USA), 1987, 502 p.

[7] Suoranta R. Amplitude domain approach to digital filtering. Theory and applications: Thesis for the degree of Doctor of Technology, Tampere (Finland, Tampere University of Technology), 1995, 199 p.

[8] Lukin V., Melnik V., Chemerovsky V., Zelensky A., Astola J., Kuosmanen P., Huttunen H., Property Analysis of Noise Suppressing L-Filters for Speckle Image Processing, Proceedings of IS&T/SPIE Symposium on Electronic Imaging: Science and Technology, San Jose (CA, USA), 1999, SPIE Vol. 3646, pp.174-184.