Invariant topological characteristics of a geometrical object.

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The work considers a method of calculation of invariant topological characteristics of a geometrical object. A geometrical object is considered as a simplicial complex. Curvilinear simplexes correspond to homeomorphic rectilinear simplexes. For each k-dimensional simplex incidence matrixes are calculated. The matrix are reduced to the canonical form. The set of incidence matrix's invariant multipliers of the simplicial complex is the characteristic of a geometrical object.

1. Introduction.

The visual perception first of all fixes the invariant, principal characteristics of an object, and then the unessential ones supplementing them with smaller details of a metric nature. This perception "stability" to the "white noise" is taken into account while describing an object when it is required to detect the major invariant characteristics of an object. For many problems "fine" metric properties of an object are unessential when topological characteristics invariant to continuous deformations of an object are important. This work is devoted to the development of invariant topological characteristics of a geometrical object.

2. Basic terms and definitions.

Let an object be represented as a set of points in the Euclidean space $\mathbf{R}^n$ (the number $\mathbf{n}$ of dimensions may be greater) not as a solid, but as a figure, allowing continuous deformation that preserves qualitative properties of an object. In this case homeomorphisms allow to eliminate unimportant details and disclose the major ones in a geometrical object [1].

Homeomorphism is a one-to one and mutually continuous to both sides mapping of one figure into another. For example rectilinear line segment and continuous arch (without self-intersections) are homeomorphic on the flat, quadrate and circle are homeomorphic, cube and tetrahedron are homeomorphic.

The set of points $\mathbf{A_0}$, $\mathbf{A_1}$, ..., $\mathbf{A_n}$ of the Euclidean space $\mathbf{R^{n+1}}$ forms $\mathbf{n}$-dimensional rectilinear simplex $\mathbf{G}^{\mathbf{n}}$ (in other words $\mathbf{A_0}$, $\mathbf{A_1}$, ..., $\mathbf{A_n}$ are vertexes of simplex $\mathbf{G^n}$), if

$$\mathbf{OA} = \sum_{i=0}^{n} \mathbf{m_i OA_i} \,,$$

where $\mathbf{O}$ is the origin of coordinates, $\mathbf{m_0}$, $\mathbf{m_1}$, ..., $\mathbf{m_n}$ are barycentric coordinates of a point $\mathbf{A}$ which is the center of gravity of points $\mathbf{A_0}$, $\mathbf{A_1}$, ..., $\mathbf{A_n}$;

$$\sum_{i=0}^{n} \mathbf{m_i = 1}, \quad \mathbf{m_i \geq 0} .$$

A zero-dimensional simplex is represented as a point, one-dimensional simplex – as a rectilinear line segment on the flat, two-dimensional simplex – as a triangle in a three-dimensional space, three-dimensional simplex – as a tetrahedron (fig.1).

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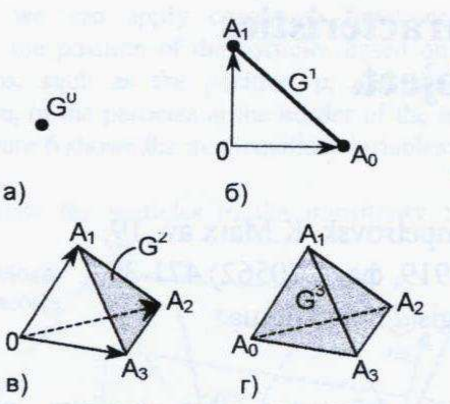

Fig.1. Examples of rectilinear simplexes: a) – zero-dimensional, b) – one-dimensional, c) – two-dimensional, d) – three-dimensional

The curvilinear simplexes are the reflexes of the rectilinear ones under homeomorphisms (fig.2)

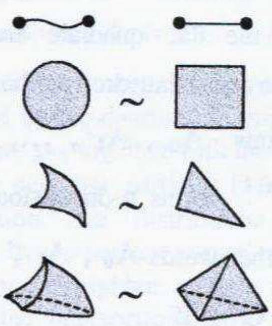

Fig.2. Examples of homeomorphic curvilinear and rectilinear simplexes.

If a set of points of **n**-dimensional simplex has one coordinate $m_i = 0$, then this set forms **(n-1)**-dimensional simplex, that is called **(n-1)**-dimensional facet of simplex G^n and is denoted as G_i^{n-1}. Simplex G^n has **(n+1)** facets of dimension **(n-1)**. If some **(n-k)** coordinates are equal to zero, then a **k**-dimensional facet of G^n simplex is considered. The number of **k**- dimensional facets in G^n simplex is equal to C_{n+1}^{k+1} - the number of combinations of **(k+1)** from **(n+1)**.

3. Constructing the matrix of incident elements characterizing a geometrical object.

Let's represent the **X** set (the object of recognition) of points of the Euclidean space as a union of finite number **N** of curvilinear simplexes having dimensions from zero to some **n** number, where

1) each point of the **X** set gets into some simplex;

2) two simplexes either have no common points or one of them is a facet of the other or they have common facet that is their intersection (fig.3).

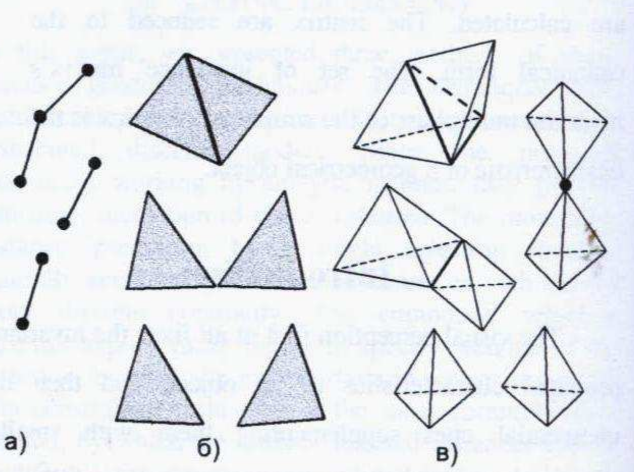

Fig.3. Variants of positional relationships of one-dimensional (a), two-dimensional (b) and three-dimensional (c) simplexes.

We will call this union a simplicial complex and denote it by **Q**.

With each curvilinear simplex some topological mapping of a rectilinear simplex to this curvilinear simplex is associated, the way the splice of two curvilinear simplexes by the common facet is corresponded by the splice of two corresponding rectilinear simplexes by some linear mapping of common facet.

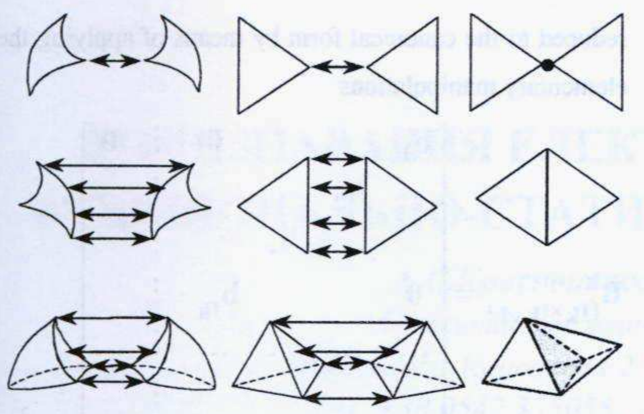

Fig. 4. Splice by zero-dimensional (a), one-dimensional (b), two-dimensional (c) facet.

When **N** is little, the **X** set is rather a "solid" body, if each simplex is considered as a solid and rotating simplexes around their common facets is allowed. With **N** increasing, the **X** set becomes more "pliable".

Let's consider the set of all **k**-dimensional simplexes of the X simplicial complex, number them, give each simplex an orientation and denote it as G_i^k, where **i** is a number, $1 \le i \le t_k$, $t_k < \infty$, t_k is the number of **k**-dimensional simplexes in **X**, $1 \le k \le n$, $n = \dim X$.

Two simplexes in the simplicial complex **Q** are called incidental if one of them is a facet of another.

The margin ∂G^k of oriented simplex G^k is the union of all it's **(k-1)**-dimensional facets taking into consideration their orientation:

$$\partial G^k = \sum_{i=0}^{k} (-1)^i G_i^{k-1}$$

For example, $\partial G^1 = A_1 - A_0$,

$$\partial G^2 = A_0A_1 + A_1A_2 - A_0A_2.$$

In this case

$$\partial G_j^{k+1} = \sum_{i=1}^{t_k} a_{ij}^k G_j^k \ ,$$

$1 \le j \le t_{k+1}$, $1 \le i \le t_k$.

The number a_{ij}^k is a coefficient of incidence of simplexes G_i^k and G_j^{k+1}. It is equal either to +1, or to -1, depending on the orientation with which the simplex G_i^k is included into the margin of the simplex G_j^{k+1} (+1, if the orientation assigned on G_i^k, coincides with the orientation induced on it by the G_j^{k+1} simplex; -1 otherwise), or to 0, if simplexes G_i^k and G_j^{k+1} are not incidental. Numbers a_{ij}^k, $1 \le j \le t_{k+1}$, $1 \le i \le t_k$, form **k**-dimensional incidence matrix of the size $t_k \times t_{k+1}$:

$$B^k = (a_{ij}^k)$$

Each simplicial complex **Q** associates with a suit of incidence matrixes: B^0, B^1,..., B^{n-1}. For example for the simplex G^2 (fig.5)

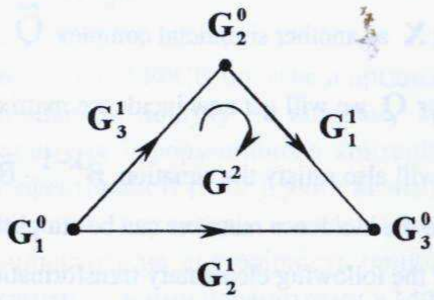

Fig. 5. Two-dimensional oriented simplex G^2

incidence matrixes look as follows:

$$B^0 = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The simplicial complex **Q** is completely and unambiguously determined by the set B^0, B^1, ..., B^{n-1} of incidence matrixes, i. e. all the properties of the **Q** complex are determined by its incidence matrixes.

4. Calculating invariant topological characteristics by means of incidence matrixes.

Since the square of the margin operator equals to zero: $\partial^2 = 0$ (the simplex margin has no margin itself), then

$$\partial\,(\partial G_j^{k+1}) = \sum_{i=1}^{t_k} a_{ij}^k\, \partial G_i^k =$$

$$= \sum_{m=1}^{t_{k-1}} \sum_{i=1}^{t_k} a_{ij}^k\, a_{mi}^{k-1}\, \partial G_m^{k-1} = 0.$$

Hence,

$$\sum_{i=1}^{t_k} a_{mi}^{k-1}\, a_{ij}^k = 0\,,\ 1 \le j \le t_{k+1}\,,\quad \le m \le t_{k-1}.$$

At the settled $\mathbf{k}$

$$B^{k-1} \cdot B^k = 0, \quad k=0, 1,\ldots, n-1.$$

This matrix equation will not change if we represent $\mathbf{X}$ as another simplicial complex $\widetilde{Q}$.

For $\widetilde{Q}$ we will get new incidence matrixes $\widetilde{B}^k$, but they will also satisfy the equation $\widetilde{B}^{k-1} \cdot \widetilde{B}^k = 0$. Therefore the incidence matrixes can be simplified by means of the following elementary transformations of simplexes.

If the G_i^k simplex is replaced by $G_i^k + G_m^k$, $i \ne m$, the matrix $\widetilde{B}^{k-1}$ is obtained from the matrix B^{k-1} by adding its m-th column to the i-th column of B^{k-1}. The matrix $\widetilde{B}^k$ is obtained from the matrix B^k by subtracting the i-th row from its m-th row. If the simplex G_i^k is replaced by $(-G_i^k)$, then the matrix $\widetilde{B}^{k-1}$ is obtained from B^{k-1} by multiplying the i-th column of B^{k-1} by (-1), and the matrix $\widetilde{B}^k$ is obtained by multiplying the i-th row of the matrix B^k by (-1). It is known [3], that any integer matrix can be reduced to the canonical form by means of applying the elementary manipulations

$$B^k_{(t_k \times t_{k+1})} = \left[\begin{array}{cccc|c} b_1 & & & 0 & 0 \\ & b_2 & & & \vdots \\ & & \ddots & & \vdots \\ 0 & & & b_{r_k} & \vdots \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & & & & 0 \end{array}\right]$$

where $r_k = \operatorname{rang} B^k$; $b_1, b_2, \ldots, b_{r_k}$ - nonzero invariant multipliers of a matrix.

Invariant multipliers are unambiguously determined by the matrix and can be calculated algorithmically. The suit of invariant multipliers of incidence matrixes of the simplicial complex $\mathbf{Q}$ is the characteristic of the geometrical object $\mathbf{X}$. Characteristics are also the numbers

$$\beta_k = t_k - r_k - r_{k-1},$$

$$(r_{-1} = r_n = 0), \quad 1 \le k \le n, \quad n = \dim X.$$

5. Conclusions.

A simple and evident method of calculating the invariant topological characteristics of a geometrical object was considered, based on representing the object as a simplicial complex and applying the incidence matrixes. Practical implementation of the described method involves cumbersome calculations, since it contains operations of transforming big matrixes.

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